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EE 364 - Spring 2026

Week 11
Supplemental Notes

Topics

① Key acronyms:

UC MOPED, $M \to P \to D$, (UEOPD)

MI, CI, WLLN (SLLN)

② Stochastic convergence

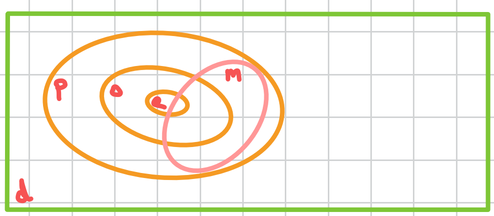
$a_n \xrightarrow{e} a$ iff $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{Z}^+ .\ \forall n \geq n_0\ |a_n - a| < \varepsilon$

trick: pick a_n and a to give e, m, p, d

	a_n	a	
e-everywhere	$X_n(\omega)$	$X(\omega)$	
m-mean square	$E[(X_n - X)^2]$	0	
p-in probability	$P[\|X_n - X\| > \varepsilon]$	0	$\forall \varepsilon > 0$
d-in distribution	$F_{X_n}(x)$ at points of continuity x	$F_X(x)$	

o - with probability one: $P(\lim_{n \to \infty} X_n = X)$

$m \to p \to d$



Estimation

Defn: Estimator $\hat{\Theta}_n$ estimates the estimand θ

function of data

parameter (constant or r.v.)

Ex: • $\hat{\Theta}_n = \bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ estimate for $\theta = \mu_X$

• $\hat{\Theta}_n = S_X = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}(X_k - \bar{X}_n)^2}$ estimate for $\theta = \sigma_X$

notation is important

Convergence of random variables

Reminder (limit of sequence) a sequence of numbers $a_1, a_2, \ldots$ converges to a ($\lim_{n\to\infty} a_n = a$) iff $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n\ \ |a_n - a| < \varepsilon$

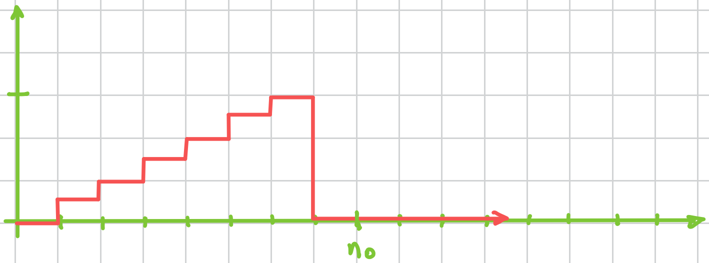
(limit of functions) a sequence of functions $f_1, f_2, \ldots$ converges pointwise to f ($\lim_{n\to\infty} f_n = f$) iff $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n\ \ |f_n(x) - f(x)| < \varepsilon$.

Garden hose analogy

Examples

① Game: heads $\longrightarrow$ win \$1.

tails $\longrightarrow$ lose all, game ends

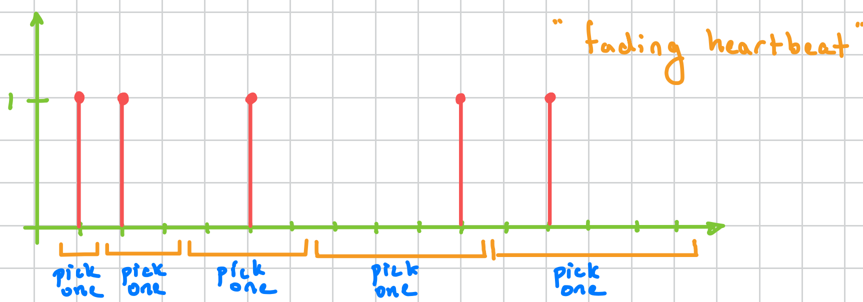


$X_n = 1\ 2\ 3\ 4\ 5\ 6\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ldots$

alternative, head $\longrightarrow$ double \$

$Y_n = 1\ 2\ 4\ 8\ 16\ 32\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ldots$

② Suppose: alive as long as heartbeat in future



$X_n = 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0 \ldots$

$P[\text{ differ } 0\ ] \leq \frac{1}{\sqrt{n}} \longrightarrow 0$

alternative: $\frac{1}{n}$ or 0

$Y_n = 1\ \tfrac{1}{2}\ 0\ 0\ \tfrac{1}{3}\ 0\ 0\ 0\ 0\ \tfrac{1}{4}\ 0\ \tfrac{1}{5}\ 0\ 0\ 0 \ldots$

Ex: Now consider the sequence of sample means

$\bar{X}_1, \bar{X}_2, \bar{X}_3, \ldots \qquad \bar{X}_n = \sum_{k=1}^{n} X_k$

① and ② Both X_n and Y_n get close to 0, but in a different "sense"

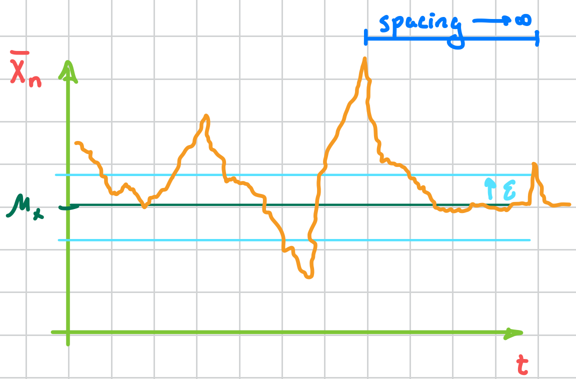
Laws of large numbers

Q: What is the long term (large n) behavior of $\bar{X}_n$?

A: $\bar{X}_n \xrightarrow{P} \mu_x$

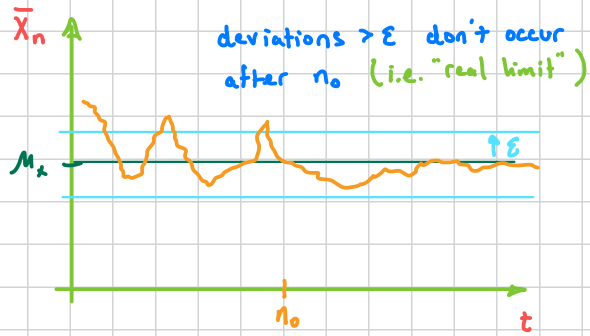
WLLN

Compare



Weak Law of Large numbers

converge in prob.



Strong law of Large numbers

converge w/ prob 1

Defn: A sequence of random variables $X_1, X_2, \dots$ are random samples if they are iid and $\sigma_x^2 < \infty$ (note: $\therefore \mu_x$ exists)

Random sample (iid, $\sigma^2 < \infty$)

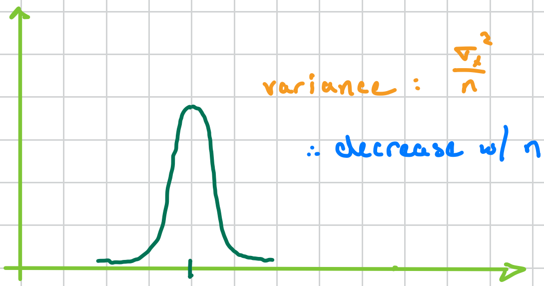
$\frac{1}{n}$ LLN: $\bar{X}_n \xrightarrow{m,o,p,d} \mu_x$ "mean to mean"

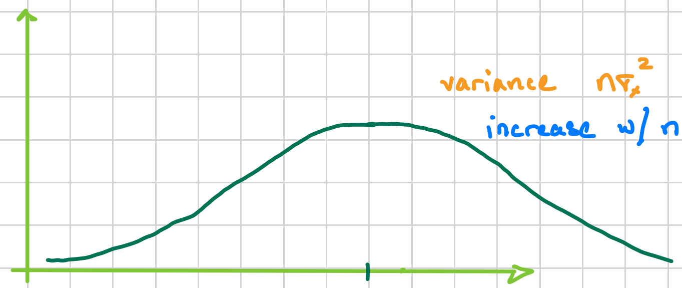
$\frac{1}{\sqrt{n}}$ CLT: $Z_n \xrightarrow{d} N(0,1)$ "standard to standard"

$$Z_n \triangleq STD(\bar{X}_n) = STD\left(\sum_{k=1}^{n} X_k\right)$$

for large n ($n \geq 30$)

$$\bar{X}_n \sim N\left(\mu_x, \frac{\sigma_x^2}{n}\right) \qquad \sum_{k=1}^{n} X_k \sim N\left(n \cdot \mu_x, n \cdot \sigma_x^2\right)$$





But: $\underline{STD}(\bar{X}_n) = \underline{STD}\left(\sum_{k=1}^{n} X_k\right)$

"Random Sampling" (r.s.)

$\longleftrightarrow$ iid $X_1, X_2, \ldots$ (and usually $\sigma_x^2 < \infty$ unless $X \sim$ Cauchy)

☆ **Thrm:** ("Sampling Statistics") If r.s. iid $X_1, \ldots X_n$ and $\sigma_x^2 < \infty$

(1) $E[\bar{X}_n] = \mu_x$

(2) $V[\bar{X}_n] = \frac{\sigma_x^2}{n}$ ← (false if $X_k \sim$ Cauchy)

Prf: ① $E[\bar{X}_n] = E\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] = \frac{1}{n}\sum_{k=1}^{n} E[X_k] \overset{\text{idnt.}}{=} \frac{1}{n}\, n \cdot \mu_x = \mu_x.$

② $V[\bar{X}_n] = V\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] \overset{\text{ind}}{=} \frac{1}{n^2}\sum_{k=1}^{n} V[X_k] = \frac{1}{n^2} \cdot n \cdot \sigma_x^2 = \frac{\sigma_x^2}{n}$

Corr: ① $E\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} E[X_k] = n \cdot \mu_x.$

② $V\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} V[X_k] = n \cdot \sigma_x^2$

Corr: $\therefore\ Z_n \overset{\Delta}{=} \mathrm{STD}\left(\sum_{k=1}^{n} X_k\right) = \dfrac{\sum_{k=1}^{n} X_k - E\left[\sum_{k=1}^{n} X_k\right]}{\sqrt{V\left[\sum_{k=1}^{n} X_k\right]}} \cdot \dfrac{\frac{1}{n}}{\frac{1}{n}}$

$$= \frac{\frac{1}{n}\sum_{k=1}^{n} X_k - \frac{1}{n} E\left[\sum_{k=1}^{n} X_k\right]}{\frac{1}{n}\cdot\sqrt{V\left[\sum_{k=1}^{n} X_k\right]}}$$

$$= \frac{\bar{X}_n - \frac{1}{n}\cdot(n\cdot\mu_x)}{\frac{1}{n}\cdot\sqrt{n}\,\sigma_x}$$

$$= \frac{\bar{X}_n - \mu_x}{\sigma_x/\sqrt{n}} = \frac{\bar{X}_n - E[\bar{X}_n]}{\sqrt{V[\bar{X}_n]}}$$

$$= \mathrm{STD}(\bar{X}_n)$$

For large n $(n \geq 30)$

$$\bar{X}_n \overset{\text{"d"}}{\approx} N\left(\mu_x, \frac{\sigma_x^2}{n}\right)$$

$$\sum_{k=1}^{n} X_k \overset{\text{"d"}}{\approx} N\left(n\cdot\mu_x,\ n\cdot\sigma_x^2\right)$$

Dividing by n (vs $\sqrt{n}$) radically changes behavior.

Say $E[X] = 0$
$\sqrt{V[X]} = 1$

$$\longrightarrow\ Z_n = \frac{\bar{X}_n - \mu_x}{\sigma_x/\sqrt{n}} = \sqrt{n}\cdot\bar{X}_n$$

$$= \sqrt{n}\cdot\frac{1}{n}\sum_{k=1}^{n} X_k$$

$$= \frac{1}{\sqrt{n}}\underbrace{\sum_{k=0}^{n} X_k}_{\text{sum}}$$

For large n $(n \gg 0)$ $\quad \bar{X}_n \approx N\left(\mu_x, \frac{\sigma_x^2}{n}\right)$

approaches a deterministic value
(thin spike)

☆ CLT applies to aggregates of data

If randomly sample (of size n) from a finite population (of size N) without replacement:

(1) still holds, but (2) becomes (3) $V[\bar{X}_n] = \frac{\sigma_x^2}{n} \cdot \underbrace{\frac{N-n}{N-1}}_{\text{correction factor}}$

2 Main Limit Theorems

$\frac{1}{n}$ ① Law of large numbers (LLN) (r.s., $\sigma_x^2 < \infty$)

$$\bar{X}_n \xrightarrow{mopd} \mu_x$$

$\frac{1}{\sqrt{n}}$ ② Central limit theorem (CLT)

$$Z_n \xrightarrow{d} Z \sim N(0,1)$$

for $Z_n = Std(\bar{X}_n) = \frac{\bar{X}_n - \mu_x}{\sigma_x/\sqrt{n}} = \frac{\sum_{k=1}^{n} X_k - n \cdot \mu_x}{\sqrt{n} \cdot \sigma_x}$

Acronym: UC MOPED ☆☆

U: uniform convergence

$$X_n \xrightarrow{u} X \text{ iff } \forall \varepsilon > 0\ \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n_0\ \left| X_n(\omega) - X(\omega) \right| < \varepsilon\ \forall x.$$

$\exists \forall$

C: Cauchy criterion

$$X_n \xrightarrow{c} X \text{ iff}: \forall \omega: \forall \varepsilon > 0\ \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n_0\ \forall m \geq n_0$$

(since $\mathbb{R}$ is "complete")

$$\left| X_n(\omega) - X_m(\omega) \right| < \varepsilon$$

M: Mean-square convergence

$$X_n \xrightarrow{m} X \text{ iff } \lim_{n \to \infty} E\left[(X_n - X)^2 \right] = 0$$

Cauchy Criterion for MS convergence

$$\text{iff } \lim_{n \to \infty} \lim_{m \to \infty} E\left[(X_n - X_m)^2 \right] = 0$$

O: Probability One ("strong" "almost surely") convergence

$$X_n \xrightarrow{o} X \text{ iff } P\left[\lim_{n \to \infty} X_n = X \right] = 1 \text{ iff } P\left[\lim_{n \to \infty} X_n \neq X \right] = 0$$

P: Convergence in Probability ("weak")

$$X_n \xrightarrow{p} X \text{ iff } \forall \varepsilon > 0\ \lim_{n \to \infty} P\left(|X_n - X| > \varepsilon \right) = 0$$

$$\text{iff } \lim_{n \to \infty} P\left(|X_n - X| \leq \varepsilon \right) = 1$$

E: Convergence Everywhere

$$X_n \xrightarrow{e} X \text{ iff } \forall \omega \in \Omega \ \forall \varepsilon > 0 \ \exists n_0 \in \mathbb{Z}^+ \ \forall n \geq n_0$$

$\forall \exists$

$$|X_n(\omega) - X(\omega)| < \varepsilon$$

D: Convergence in Distribution ("in law")

$$X_n \xrightarrow{d} X \text{ iff } \lim_{n \to \infty} F_{X_n}(x) = F_X(x) \text{ at points of continuity}$$

$\underline{O}$: convergence w/ probability 1 (almost surely / almost everywhere)

$$X_n \xrightarrow[\text{(a.e.) (a.s.)}]{o} X \quad \longleftrightarrow \quad P\left(\lim_{n\to\infty} X_n = X\right) = 1$$

Ex: "Fading heartbeat" $X_n \longrightarrow 0$

$\cancel{X_k = 1}$ $\quad X_k = \frac{1}{k} \quad$ $\cancel{X_k = k}$

Ex: Add \$1, vs. Add \$ k, vs. double \$

$$P[X_k = 0] = \frac{1}{2^k}$$

$$\lim_{n\to\infty} E[(X_n - X)^2] = \lim_{n\to\infty} E[X_n^2] = \lim_{n\to\infty}\left(0^2 \cdot \left(1 - \frac{1}{2^n}\right) + (2^n)^2 \frac{1}{2^n}\right)$$

$$= \lim_{n\to\infty} 2^n = \infty$$

$\underline{m}$: Convergence in mean square

$$X_n \xrightarrow{m} X \quad \longleftrightarrow \quad \lim_{n\to\infty} E[(X_n - X)^2] = 0$$

expected value

"average vs. possible" (probable)

Not robust. It can "blow up" $(=\infty)$ and it gives too much importance to outliers. (even if very rare)

see example above

optional

$\underline{C}$: $\{a_n\}$ is Cauchy $\longleftrightarrow$ $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n_0, m \geq n_0 \quad |a_n - a_m| < \varepsilon.$

note $\longleftrightarrow$ $\quad \therefore a_n \longrightarrow a$ (since $\mathbb{R}$ complete)

Cauchy Criterion: $X_n \xrightarrow{m} X \longleftrightarrow \lim_{n\to\infty} \lim_{m\to\infty} \underbrace{E[(X_n - X)^2]}_{a_n} = \underbrace{0}_{a}$

Useful because shows convergence a_n without knowing (guessing) a

D: Convergence in distribution

$X_n \xrightarrow{d} X \longleftrightarrow \forall$ continuity points x: $\lim_{n\to\infty} \underbrace{F_{X_n}(x)}_{a_n} = \underbrace{F_X(x)}_{a}$

discrete $\rightarrow$ continuous

$\therefore$ jumps OK

Ex: CLT iid, $\sigma^2 < \infty$

$Z_n = STD(\bar{X}_n) = STD\left(\sum_{k=1}^{n} X_k\right) \xrightarrow{d} Z \sim N(0,1)$

Ex: Poisson law $b(n,p) \xrightarrow{d} P(\underbrace{\lambda}_{=n \cdot p})$

Thm: ("Markov Inequality", MI) If $X \geq 0$ and $c \in \mathbb{R}^+$ and $E[X]$ exists $P[X \geq c] \leq \frac{E_X[X]}{c}$

Prf: (continuous case)

$E_X[X] = \int_0^\infty x \cdot f_X(x)\, dx$

$= \int_0^c \underbrace{x \cdot f_X(x)\, dx}_{\geq 0} + \int_c^\infty x \cdot f_X(x)\, dx$

since $x \geq 0$ and f: pdf

$\geq \int_c^\infty x \cdot f_X(x)\, dx.$ since $x \geq 0$

$= \int_{x \geq c} x \cdot f_X(x)\, dx \geq \int_{x \geq c} c \cdot f_X(x)\, dx$

$= c \cdot \int_c^\infty f_X(x)\, dx = c \cdot P[X \geq c]$

$\therefore P[X \geq c] \leq \frac{E_X[X]}{c}$

QED

Thrm: ("Chebyshev Inequality", CI) If $\sigma_X^2 < \infty$:

for $\forall \varepsilon > 0$: $P\left[|X - \mu_X| > \varepsilon\right] \leq \dfrac{V_X[X]}{\varepsilon^2}$

Prf: Pick $\varepsilon > 0$:

$$P\left[|X-\mu| > \varepsilon\right] = P\left[(X-\mu)^2 > \varepsilon^2\right]$$

$$\overset{M.I.}{\leq} \frac{E_X\left[(X-\mu)^2\right]}{\varepsilon^2} = \frac{V_X[X]}{\varepsilon^2}$$

QED

Put $\varepsilon = k\sigma$ for $k \in \mathbb{Z}^+$ $\quad \therefore P\left[|X-\mu| > k\sigma\right] \leq \frac{1}{k^2}$

$$\therefore 1 - P\left[|X-\mu_X| > k\sigma\right] \geq 1 - \frac{1}{k^2}$$

$$\therefore \boxed{P\left[|X-\mu_X| \leq k\sigma^2\right] \geq 1 - \frac{1}{k^2}}$$

Ex: $k = 3$. Probability of r.v. X being within 3-std deviations of the mean?
(assume $\sigma^2 < \infty$)

$$P\left(|X-\mu| \leq 3\sigma_X\right) \geq 1 - \frac{1}{3^2} = \frac{8}{9}$$

for any pdf

bound vs. exact.

$$\therefore \text{also, } P\left(|X-\mu| \geq 3\sigma_X\right) \leq \frac{1}{9}$$

Compare: $X \sim N(0,1)$

$$P\left(|X-0| \leq 3\right) \approx 0.997$$

std. normal.